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INSTABILITY IN THE CRITICAL CASE OF A PAIR OF PURE IMAGINARY ROOTS FOR A CLASS OF SYSTEMS WITH AFTEREFFECT†

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The stability of motion of a system described by Volterra integrodifferential equations is investigated in the critical case when the characteristic equation has a pair of pure imaginary roots. Conditions for instability, analogous to the well-known conditions from the theory of differential equations [1], are derived. (A similar result was established previously in [2] for integro-differential equations of simpler structure with integral kernels of exponential-polynomial type.) For the proof, several manipulations are used to simplify the original equation and, in particular, to reduce the linearized equation to the form of a differential equation with constant diagonal matrix. (An analogous approach was used to analyse instability for Volterra integrodifferential equations in the critical case of one zero root in [3, 4].) As an example, the sign of the Lyapunov constant in the problem of the rotational motion of a rigid body with viscoelastic supports is calculated.

1. We will consider a system with aftereffect, whose perturbed motion in the neighbourhood of the motion being investigated is described by the equation

$$\frac{dx}{dt} = Ax + \int_0^t K(t-s)x(s)ds + F(x, \tilde{y}, t), \quad x \in R^n, \quad \tilde{y} \in R^m \tag{1.1}$$

where A is a constant $n \times n$ matrix, and the $n \times n$ matrix $K(t) \in C$ is defined on the set $I = \{t \in R: t \geqslant 0\}$ and satisfies the inequality

$$\|K(t)\| \leqslant C\exp(-\beta t), \quad C, \beta = \text{const} > 0 \tag{1.2}$$

The vector-valued function $F(x, \tilde{y}, t)$: $B_2(x, \tilde{y}) \times I \to R^n$ in (1.1), where $B_2(x, \tilde{y}) = \{x \in R^n, \tilde{y} \in R^m: \|x\| < R_1, \|\tilde{y}\| < R_2\}$ for given $R_i > 0$ $(i = 1, 2)$, is assumed to be holomorphic in x and $\tilde{y}$; moreover, it is assumed that the coefficients of its power series expansion are either continuous and tend exponentially to constants as $t \to +\infty$, or are constants. The functional $\tilde{y}$ has the form

$$\tilde{y} = \int_0^t \tilde{K}(t-s)\phi(x(s), s)ds \tag{1.3}$$

$$\phi(x, t): B_1(x) \times I \to R^k, \quad B_1(x) = \{x \in R^n: \|x\| < R_1\}$$

where $\phi(x, t)$ is a vector-valued function, holomorphic in x, with expansion coefficients of the same type as $F(x, \tilde{y}, t)$, and $K(t) \in C$ is an $m \times k$ matrix given for $t \in I$ such that

$$\|\tilde{K}(t)\| \leqslant \tilde{C}\exp(-\varkappa t), \quad \tilde{C}, \ \varkappa = \text{const} > 0 \tag{1.4}$$

We will assume that the functions F and ϕ are such that, after x has been replaced by εx (ε = const), this substitution also including Eq. (1.3), the expansion of F in a series of powers of ε begins with terms of not less than the second order.

The Cauchy problem can be considered for Eq. (1.1)–(1.4) and the Lyapunov stability of the trivial solution can be investigated with respect to disturbance of the initial conditions $x(0)$.

In what follows we shall use the following notation.

If a function $\psi(t)$ satisfies an inequality of the following type for $t \in I$

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$$\|\psi(t)\| \leq c\exp(\gamma t), \quad c = \text{const} > 0$$

then we write $\psi(t) \in e_1(\gamma)$, that is, $\psi(t)$ belongs to the class $e_1(\gamma)$.

Similarly, if $\psi_1(t, s)$ is a function defined on the set $J = \{(t, s) \in R^2\colon 0 \leq s \leq t < +\infty\}$, satisfying the inequality

$$\|\psi_1(t,s)\| \leq c\exp[\gamma(t-s)]$$

then we write $\psi_1(t, s) \in e_2(\gamma)$.

Let $K^*(\lambda)$ be the Laplace transform of the matrix $K(t)$. By (1.2), the characteristic equation for (1.1)

$$\det(\lambda E_n - A - K^*(\lambda)) = 0 \tag{1.5}$$

exists for Re $\lambda \geq -\beta$ and the determinant in (1.5) is analytic for Re $\lambda > -\beta$. We shall assume that Eq. (1.5) has a finite number of roots in the half-plane Re $\lambda > -\beta$, say λ_j' ($j = 1, \ldots, N$), numbered in order of increasing real parts, with Re $\lambda_j' < 0$ ($j = 1, \ldots, N-2$) and $\lambda_{N-1}' = i\omega$, $\lambda_N' = -i\omega$, $\omega > 0$. Suppose that λ_l ($l = 1, \ldots, n$) are the characteristic exponents of the solutions of the linearized equation (1.1) such that

$$-\beta < \lambda_1 \leq \lambda_2 \leq \ \ldots \ \leq \lambda_{n-2} < \lambda_{n-1} = \lambda_n = 0 \tag{1.6}$$

and that all the roots of the characteristic equation corresponding to λ_l ($l = 1, \ldots, n$) are simple and Re $\lambda_s' < \lambda_1$ ($s = 1, \ldots, N-n$); some of them may be complex conjugates: $\lambda_s = \mu_s + i\omega_s$, $\lambda_{s+1} = \mu_s - i\omega_s$ ($s = 1, \ldots, p$). Then the resolvent of the linearized equation (1.1) may be expressed as [5]

$$R(t) = \sum_{l=N-n+1}^{N} p_l \exp(\lambda_l' t) + R_1(t), \quad t \in I, \quad p_l = \text{const} \tag{1.7}$$

where the $n \times n$ matrix $R_1(t) \in C^1$ is such that $R_1(t) \in e_1(-\beta_1)$, where $\beta_1 > 0$ is a constant satisfying the inequality $-\beta < -\beta_1 < \lambda_1$. We shall assume that for some $\beta' \geq \beta_1$

$$dR_1(t)/dt \in e_1(-\beta') \tag{1.8}$$

2. We will now perform a series of transformations that will enable us to single out critical variables. We introduce a fundamental solution matrix $X'(t)$ of the linearized equation (1.1) and suppose it to be normal in the Lyapunov's sense [1]. If $x_l'(t)$ ($l = 1, \ldots, n$) are fundamental solutions (columns of $X'(t)$), then the characteristic exponents satisfy the equalities $\chi(x_j'(t)) = \lambda_j$ ($j = 1, \ldots, n-2$) and

$$\begin{aligned} x_{n-1}'(t) &= 2(a\cos\omega t - b\sin\omega t) + x_{n-1}''(t) \\ x_n'(t) &= 2(b\cos\omega t + a\sin\omega t) + x_n''(t) \end{aligned} \tag{2.1}$$

where a and b are constant vectors and $\chi(x'_j(t)) \leq \lambda_{n-2}$, $k = n-1, n$. Define a function

$$d(t) = \exp\left(-\sum_{j=1}^{n-2} \lambda_j t\right)\det X'(t) \tag{2.2}$$

which, as follows from the structure of the fundamental solutions, may be expressed as $d(t) = d_0 + d_1(t)$, where $d_0 = \text{const}$ and $d_1(t) \in e_1(\lambda_{n-2})$. Let us assume that for $t \in I$ this function satisfies the condition

$$|d(t)| \geq d' > 0, \quad d' = \text{const} \tag{2.3}$$

Let us consider the basis conjugate to $x_l'(t)$, say $y_l'(t)$, whose vectors are the rows of a matrix $Y'(t) = (y_{lj}''(t))$ such that $Y'(t)X'(t) = E_n$. Define a fundamental solution matrix $X(t-s)$ ($X(0) = E_n$) of the linearized equation (1.1) with lower limit of integration s, with whose help the general solution of Eq. (1.1) may be expressed in terms of the Cauchy integral formula [6].

It follows from the structure of the general solution (1.7) and from (2.3) that the linearized equation (1.1) is regular in Lyapunov's sense. Consequently, we have the equalities $\chi(y_l'(t)) = -\lambda_l$ ($l = 1, \ldots, n$) and

$$\begin{aligned} y'_{lj}(t) &= \exp(-\lambda'_l t)(c_{lj} + y''_{lj}(t)) \\ y'_{n-1j}(t) &= \delta_j(b)\cos\omega t + \delta_j(a)\sin\omega t + y''_{n-1j}(t) \\ y''_{nj}(t) &= -\delta_j(a)\cos\omega t + \delta_j(b)\sin\omega t + y''_{nj}(t) \end{aligned} \tag{2.4}$$

where c_{lj} are (real or complex) constants, $\delta_j(a)$, $\delta_j(b)$ are real constants and $y''_{kj}(t) \in e_1(\alpha)$, $\alpha < 0$ $(k, j = 1, \ldots, n;\ l = 1, \ldots, n-2)$. We make the change of variables

$$y_l = x_l, \quad y_k = \sum_{j=1}^{n} y'_{kj}(t)x_j, \quad l = 1,\ldots,n-2, \quad k = n-1, n \tag{2.5}$$

where the coefficients are continuous and bounded for $t \in I$; provided that

$$\begin{aligned} \delta(t) &= |y'_{n-1n-1}(t)y'_{nn}(t) - y'_{n-1n}(t)y'_{nn-1}(t)| = \\ &= |\delta_0 + \delta_1(t)| \geqslant \delta'_0 > 0, \quad \delta_0, \delta'_0 = \text{const}, \quad \delta_1(t) \in e_1(\alpha)\ (\alpha < 0) \end{aligned} \tag{2.6}$$

this transformation is of the Lyapunov type. Changing to complex-conjugate variables

$$w_{n-1} = y_{n-1} + iy_n, \quad w_n = y_{n-1} - iy_n \tag{2.7}$$

and using (2.6), we deduce the following formulae for the transformation inverse to (2.5)

$$x_s = \sum_{k=n-1,n} W_{sk}(t)\exp(\pm i\omega t)w_k + \sum_{j=1}^{n-2} Y_{sj}(t)y_j, \quad s = n-1, n$$

$$W_{sk}(t) = W_{sk}^{(0)} + W_{sk}^{(1)}(t), \quad Y_{sj}(t) = Y_{sj}^{(0)} + Y_{sj}^{(1)}(t); \quad W_{sk}^{(0)},\ Y_{sj}^{(0)} = \text{const}$$

$W_{sk}^{(1)}(t)$, $Y_{sj}^{(1)}(t) \in e_1(-\gamma)$ for some $\gamma > 0$.

The plus sign is taken for $k = n$ and the minus for $k = n - 1$.

3. We now transform the subsystem for the non-critical variable $y = \text{col}(y_1, \ldots, y_{n-2})$. To that end we introduce a Lyapunov-normal fundamental solution matrix $X'_2(t)$ by deleting the $(n-1)$th and nth rows and columns. In the same way, we derive from the matrix $X(t-s)$ a fundamental matrix $X_2(t-s)$ $(X_2(0) = E_{n-2})$ for this subsystem.

Let $\Lambda'_2 = \text{diag}(\lambda'_{N-n+1}, \ldots, \lambda'_{N-2})$, where $\operatorname{Re}\lambda'_{N-n+l} = \lambda_l$ $(l = 1, \ldots, n-2)$. Let us assume that for $t \in I$

$$|\det(X'_2 t)\exp(-\Lambda'_2 t))| \geqslant \delta'_2 > 0, \quad \delta'_2 = \text{const} \tag{3.1}$$

Note that the determinant in this inequality tends exponentially to a constant as $t \to +\infty$.

We introduce a matrix $Y'_2(t)$ such that $Y'_2(t)X'_2(t) = E_{n-2}$ and make the substitution

$$z = \exp(\Lambda'_2 t)Y'_2(t)y \tag{3.2}$$

with coefficients that are bounded and continuous for $t \in I$ and tend to constants as $t \to +\infty$. After the transformations (2.5), (2.7) and (3.2) have been applied we obtain, using Lemma 1 of [3], equations analogous to (2.2) and (3.4) of [2]; of these equations, we will write here only those for the critical variables

$$\begin{aligned} \frac{dw_k}{dt} &= \int_0^t \sum_{j=1}^{n} (\varphi_{n-1j}(t,s) \pm i\varphi_{nj}(t,s))F'_j(z(s), w(s), \hat{y}(s), s)ds + \\ &+ \sum_{j=1}^{n} (y'_{n-1j}(t) \pm iy'_{nj}(t))F'_j(z, w, \hat{y}, t), \quad k = n-1, n; \quad w = \text{col}(w_{n-1}, w_n) \end{aligned} \tag{3.3}$$

where $\hat{y}(t)$ is the integral (1.3) transformed to the variables z, w and the functions F'_j are the components of the vector F in (1.1), transformed to the variables z, w. The upper sign in (3.3) corresponds to

$k = n - 1$. In Eqs (3.3)

$$\varphi_{kj}(t,s) = \frac{\partial}{\partial t}\left(\sum_{l=1}^{n} y'_{kl}(t)x_{lj}(t-s)\right), \quad k = n-1, n \tag{3.4}$$

Equations (3.3), as well as the equations for the non-critical variables, corresponding to the complex-conjugate elements of the matrix Λ'_2, are complex conjugate. By (2.4), we have

$$y'_{n-1j}(t) \pm iy'_{nj}(t) = c'_j \exp(\pm i\omega t) + \tilde{y}'_j(t), \quad c'_j = \text{const}, \quad \tilde{y}'_j(t) \in e_1(-\gamma), \; \gamma > 0$$

It can also be shown, using (3.4) and the relationship between $y'_{ki}(t)$ and $x_{ij}(t)$ and performing the necessary calculations, that

$$\varphi_{n-1j}(t,s) + i\varphi_{nj}(t,s) = \exp(i\omega t)\Phi_j(t-s) + \tilde{\Phi}_j(t,s), \quad \Phi_j(t) \in e_1(-\gamma) \tag{3.5}$$

where $\tilde{\Phi}_j(t, s)$ is the sum of terms of the form $\varphi_1(t)\varphi_2(t, s)$, with $\varphi_1(t) \in e_1(-\gamma_1)$, $\varphi_2(t, s) \in e_2(-\gamma_2)$ for $\gamma_1 > 0$, $\gamma_2 > 0$.

All the coefficients $\xi(t)$ of terms in (3.3) and in the subsystem for the non-critical variables that depend only on z_l ($l = 1, \ldots, n-2$) and are outside the integral sign have the structure $\xi(t) = \xi_0 + \xi_1(t)$, where ξ_0 - const, $\xi_1 \in e_1(-\gamma)$ for some $\gamma > 0$, and all the integral kernels belong to the class $e_2(-\gamma)$.

The scheme of the subsequent discussion is more or less a repetition of the proof presented in [2]. In particular, integration by parts and a substitution of the type

$$u = z + U_{4m}(w,t) + \sum_{sm(0,1)=1} w_{n-1}^k w_n^l \int_0^t N^{m(0,1)}(t,s) w_{n-1}^{kl}(s) w_n^{l1}(s) u(s) ds + U'_{2m+1}(w,t)$$

where k, l, kl and $l1$ are non-negative integers, $m(0, 1)$ is the set of these numbers, $sm(0, 1)$ is their sum, $U_{4m}(w, s)$ is a polynomial in w, of degree $4m$, with continuous bounded coefficients, and $U'_{2m+1}(w, t)$ is a finite sum of integral terms (of the indicated type) of degree greater than two, linear in u, containing multiple integrals with continuous kernels of the class $e_2(-\gamma)$ for $\gamma > 0$; all terms depending only on the variables w_{n-1} and w_n up to some order $4m$ inclusive may be successively excluded from the equation for the non-critical variables, as can integral terms that are linear in a non-critical variable of order up to and including $2m + 1$. We write the equation, thus transformed, as

$$du / dt = \Lambda'_2 u + U(u, w, t)$$

where the integral operator U has the properties described above.

After a series of simplifying transformations, enabling us to reduce terms of order up to $2m + 1$ on the right of Eqs (3.3) to an autonomous form, these equations become

$$dw'_j / dt = \sum_{k=1}^{m} C_j^{(k)} r^{2k} w'_j + \phi_j(u, w', t) \tag{3.6}$$

$$w' = \text{col}(w'_{n-1}, w'_n), \quad C_j^{(k)} = \text{const}, \quad r^2 = w'_{n-1} w'_n$$

where w' is a new critical variable, $\phi_j(u, w', t)$ is an integral operator such that the expansion in powers of ε for $\phi_j(\varepsilon u, \varepsilon w', t)$ begins with terms of the second order in ε and all terms of order up to and including $2m + 1$ vanish when $u = 0$. Note that Eqs (3.6) with $j = n - 1$ and $j = n$ are complex conjugates.

On the basis of (3.6), we set up the real equation

$$r\frac{dr}{dt} = \sum_{k=1}^{m} g_{2k+1} r^{2k+2} + R^{(3)}(\upsilon, \upsilon', t) + R^{(2m+3)}(\upsilon, \upsilon', t), \quad g_{2k+1} = \text{const} \tag{3.7}$$

where $\upsilon = \text{col}(\upsilon_1, \ldots, \upsilon_{n-2})$ and $\upsilon' = \text{col}(\upsilon_{n-1}, \upsilon_n)$ are vectors of real variables corresponding to u and w', and $R^{(2m+3)}$ are real integral operators such that $R^{(3)}(\varepsilon\upsilon, \varepsilon\upsilon', t)$ is a polynomial in ε of degree $2m + 2$ that begins with third-order terms, such that $R^{(3)}(0, \upsilon', t) \equiv 0$, and the expansion of $R^{(2m+3)}(\varepsilon\upsilon, \varepsilon\upsilon', t)$ in powers of ε begins with terms of order $2m + 3$.

Suppose that $g_3 = \ldots = g_{2m-1} = 0$ and $g_{2m+1} > 0$ in Eq. (3.7).

As in [2, 3], one can now use Chetayev's instability theorem [7, 8], which is true for integrodifferential equations of the type considered, to prove that the unperturbed motion is unstable.

Theorem. Suppose that the characteristic equation (1.5) for Eq. (1.1)–(1.4) has a finite number of roots in the half-plane Re $\lambda > -\beta$, say λ'_j ($j = 1, \ldots, N$), where Re $\lambda'_j < 0$ ($j = 1, \ldots, N-2$) and $\lambda'_{N-1} = i\omega$, $\lambda'_N = -i\omega$; suppose moreover that all the roots λ'_s ($s = 1, \ldots, n$) corresponding to characteristic exponents λ_s (1.6) are simple and Re $\lambda'_l < \lambda_1$ ($l = 1, \ldots, N-n$). Let conditions (1.8), (2.3), (2.6) and (3.1) hold.

Then, if the first non-zero constant in Eq. (3.7) is $g_{2m+1} > 0$, the trivial solution of Eq. (1.1)–(1.4) is unstable.

4. We will now investigate the stability of the equilibrium position in an example analogous to that considered in [9]. The rigid body in this example is a shaft AB whose mass distribution is the same in each cross-section. Rigidly fastened to the ends of the shaft are two viscoelastic bodies OA and BO_1 (each is a shaft of unit length and negligibly small mass), whose ends are fixed. The entire system can rotate about the axis OO_1, which is assumed to be undeformable. Let ϑ be the angle of rotation of the shaft, r the distance of the centre of mass of the shaft from the axis OO_1, mg its mass and J its moment of inertia about the axis OO_1. This rigid body is moving in a uniform gravitational field under the action of the viscoelastic forces exerted on it at its ends A and B by the bodies OA and BO_1. The torque M of these forces is assumed to have the same form as in [10, 11], on the assumption that the stress–strain relationship is given by a Volterra–Fréchet series of which only the first terms affecting the conditions derived below are retained, that is

$$M = -k\vartheta + \int_0^t K'(t-s)\vartheta(s)ds + \int_0^t\int_0^t\int_0^t \tilde{K}(t-u, t-v, t-w)\vartheta(u)\vartheta(v)\vartheta(w)du\,dv\,dw \tag{4.1}$$

(k is the modulus of elasticity for twisting and $K'(t)$ and $\tilde{K}(t_1, t_2, t_3)$ are relaxation kernels). Let us assume [11, p. 606] that

$$\tilde{K}(t-u, t-v, t-w) = K''(t-u)K''(t-v)K''(t-w)$$

We will investigate the stability in rotational motion of the equilibrium position of the rigid body when its centre of mass is in its upper position, $\vartheta = 0$. The equations of perturbed motion may be written as

$$\frac{d\vartheta}{dt} = \vartheta_1, \quad \frac{d\vartheta_1}{dt} = -K\vartheta + \int_0^t K_1(t-s)\vartheta(s)ds - m_1\vartheta^3 + \tilde{y}^3 + o(\vartheta^5) \tag{4.2}$$

$$K = \frac{k - mgr}{J}, \quad K_1(t) = \frac{K'(t)}{J}, \quad K_2 = \frac{K''(t)}{J^{1/3}}, \quad m_1 = \frac{mgr}{3!J}, \quad \tilde{y} = \int_0^t K_2(t-s)\vartheta(s)ds$$

Suppose that the kernel $K''(t)$ satisfies an estimate of the form (1.4). Assume that the characteristic equation for (4.2) has two pure imaginary roots $\pm i\omega$, the remaining roots having negative real parts and satisfying the conditions of the theorem.

After suitable calculations, we see that the sign of the constant g_3 is determined by that of the quantity

$$g'_3 = \operatorname{Re}\left\{\left[-m_1 + \left(\int_0^\infty K_2(\tau)e^{i\omega\tau}d\tau\right)^2 \int_0^\infty K_2(\tau)e^{-i\omega\tau}d\tau\right]\left[(a_1 - ib_1)\int_0^\infty \Phi_2(s)ds + \frac{i}{2\omega}\right]\right\} \tag{4.3}$$

where a_1 and b_1 are the components of the vectors a and b defined in (2.1), and $\Phi_2(s) = \Phi_2^{(1)}(s) + i\Phi_2^{(2)}(s)$ is the function occurring in representation (3.5).

If $g'_3 > 0$, our theorem implies that the equilibrium is unstable.

If the kernel $K(t)$ has an exponential-polynomial structure, the function $\Phi_2(t)$ can be evaluated explicitly, using the well-known general solution of the linearized equation. Thus suppose that $K_1(t)$ has the form

$$K_1(t) = Q_1\exp(-\gamma_1 t) + Q_2\exp(-\gamma_2 t)$$

where the constants Q_i and γ_i ($i = 1, 2$) satisfy the inequalities

$$Q_1 > 0, \quad Q_2 < 0, \quad Q_1 \geqslant |Q_2|, \quad \gamma_1 > \gamma_2 > 0 \tag{4.4}$$

Under these conditions, the characteristic equation

$$\Phi(\lambda) \equiv \lambda^2 + K - \frac{Q_1}{\lambda+\gamma_1} - \frac{Q_2}{\lambda+\gamma_2} = 0$$

has a pair of pure imaginary roots $\pm i\omega$, where

$$\omega^2 = K - \chi_0, \quad \chi_0 = \frac{Q_1 + Q_2}{\gamma_1 + \gamma_2}$$

and two roots λ'_1 and λ'_2 with negative real parts

$$\lambda'_{1,2} = -\frac{1}{2}(\gamma_1 + \gamma_2) \pm \left[\frac{1}{4}(\gamma_1 + \gamma_2)^2 - \gamma_1\gamma_2 - \chi_0\right]^{1/2}$$

provided that the following relation exists between the parameters of the system

$$K = \gamma_1\gamma_2 + \chi_0^{-1}(\chi_0^2 - Q_1\gamma_2 - Q_2\gamma_1)$$

Suppose, for simplicity, that λ'_1 and λ'_2 are real numbers. Then the solution of the linearized equation of perturbed motion may be written in the form

$$\vartheta(t) = (\alpha_1^2 + \alpha_2^2)^{-1}[(\alpha_1 - i\alpha_2)\exp(i\omega t)(i\omega\vartheta(0) + \dot{\vartheta}_1(0)) +$$
$$+(\alpha_1 + i\alpha_2)\exp(-i\omega t)(-i\omega\vartheta(0) + \vartheta_1(0))] + \sum_{k=1,2} \exp(\lambda'_k t)(\lambda'_k\vartheta(0) + \vartheta_1(0)) / \Phi'(\lambda'_k) \quad (4.5)$$

where the constants α_1 and α_2 are determined by the following relation (the prime indicates a derivative of Φ)

$$\Phi'(i\omega) = \alpha_1 + i\alpha_2$$

Using (4.5), we can calculate the functions $\Phi_2^{(k)}(t)$ $(k = 1, 2)$, which can be shown to satisfy the identity

$$a_1\Phi_2^{(1)}(t) + b_1\Phi_2^{(2)}(t) \equiv 0$$

Then the sign of g'_3 (4.3) will be the same as that of the quantity g''_3 defined by

$$g''_3 = -\phi_0 \int_0^\infty K_2(s)\sin(\omega s)ds \quad (4.6)$$

$$\phi_0 = 2\int_0^\infty (-b_1\Phi_2^{(1)}(s) + a_1\Phi_2^{(2)}(s))ds + \omega^{-1}$$

To compute the constant ϕ_0, we have the following formula

$$\phi_0 = \frac{1}{\omega}\left(1 - \sum_{k=1,2} \frac{\omega^2 + \lambda_k'^2}{\lambda'_k\Phi'(\lambda'_k)}\right)$$

Suppose, for example, in accordance with (4.4), that $\gamma_1 = 3\gamma_0$, $\gamma_2 = \gamma_0$, $Q_1 = 2\gamma_0^3$, $Q_2 = -\gamma_0^3$, where $\gamma_0 > 0$. In that case, $\phi_0 = 12\gamma_0/(13\sqrt{7}) > 0$, and the instability condition implies that the integral in the formula (4.6) for g''_3 is negative.

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